

**The Renormalization Group and the Dynamics of Genetic Systems****Christopher R. Stephens**<sup>1\*</sup>

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In this brief article I show how the notion of coarse graining and the Renormalization Group enter naturally in the dynamics of genetic systems, in particular in the presence of recombination. I show how the latter induces a dynamics wherein coarse grained and fine grained degrees of freedom are naturally linked as a function of time leading to a hierarchical dynamics that has a Feynman-diagrammatic representation. I show how this coarse grained formulation can be exploited to obtain new results.

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**1 Introduction**

The Renormalization Group (RG), in its many diverse guises, has proved to be an immensely powerful and useful tool in the treatment of systems with many degrees of freedom; with applications covering a huge gamut, from relativistic quantum field theory to the asymptotics of differential equations. In this short article I discuss another arena where the RG appears in a very natural way - genetic dynamics.

By genetic dynamics I mean the dynamics of string-, or tree-like objects whose evolution is governed by a set of genetic operators, the most common of which are selection, mutation and recombination. Selection and mutation have been extensively studied by physicists (see for example [1]). Recombination however remains relatively untouched, although it has been extensively studied in biology (see for example [2]). The chief areas of interest are population genetics, and associated fields, and evolutionary computation. In both fields of interest one may be dealing with many, many degrees of freedom and hence the normal RG motivation of reducing degrees of freedom is valid. For instance, a typical protein has  $O(10^4)$  aminoacids.

Here, I will concentrate more on showing that recombination naturally induces a coarse graining and the subsequent dynamics possesses a hierarchical structure wherein genetic configurations are related in the past to more coarse grained genetic “building blocks”. I show that such coarse grainings have a RG structure and lead to new results and insights that can be, and have been, profitably used - principally in evolutionary computation. Although the flavour of this article is quite different many of the results and more details can be found in the following articles [3–7].

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## 2 Genetic Dynamics

Consider the dynamics of a population  $\mathcal{M}(t) \equiv \{c_i(t)\} \subset \mathcal{G}$  of strings of equal length<sup>2</sup>,  $N$ , where  $\mathcal{G}$  is the configuration space of string states and  $\{c_i(t)\}$  is the set of “genotypes” present in the population at time  $t$ . For simplicity we will assume binary bits, though nothing we shall present depends on this fact. We may naturally represent  $\mathcal{G}$  as a  $N$ -dimensional hypercube, a natural metric being the Hamming distance, strings associated with adjacent vertices being Hamming distance one apart. The  $N$  string loci form a complete orthonormal basis for the hypercube.

The population evolves in discrete time under the action of an evolution operator  $\mathcal{H}$ , the action of which depends on the specific “genetic” operators involved. Here, we will consider the three canonical operators - selection, mutation and recombination - whereupon we have

$$\mathcal{M}(t+1) = \mathcal{H}(\{f_i\}, \{\mathcal{M}(t)\}, \{p_k\}, t) \mathcal{M}(t) \quad (1)$$

In this case  $\mathcal{H}$  depends on the reproductive fitness landscape,  $\{f_i\}$ , the population  $\{\mathcal{M}(t)\}$  and the set of parameters,  $\{p_k\}$ , that govern the other genetic operators; e.g. mutation and recombination probabilities. For selection  $P_i(t+1) \equiv P'_i(t) = F_{ij}P_j(t)$ , where  $F_{ij}$  is the fitness matrix and  $P_i(t)$  is the probability of finding the string  $c_i$  at time  $t$ . One of the most widely used selection schemes is proportional selection where the dynamics is given by  $F_{ij} = (f_i/\bar{f}(t))\delta_{ij}$ , where  $\bar{f}(t)$  is the average population fitness. It is usually considered as a unary operator.

Mutation is also a unary operator and, typically, is such that every string bit flips to its complement with probability  $p_m$  every generation. Recombination, in distinction, is a binary operator (although higher cardinality can be considered) and is such that a “child” string is formed by taking a certain number of bits from one “parent” string and the complement from another “parent” string. For example, one can form 1111 from parents 1010 and 0101 by taking the first and third bits from 1010 and the complementary second and fourth bits from 0101. One may specify the bits taken from the first parent using a recombination “mask”,  $m$ . In the above example the recombination mask is 1010 which signifies take bits one and three from the first parent (specified by the position of the ones) and two and four from the second.

The resultant dynamical equation describing the evolution of the probability distribution for this system is

$$P_i(t+1) = \mathcal{P}_{ij}P_c^j(t) \quad (2)$$

where  $P_c^i(t)$  is the probability to find strings of type  $c_i$  after selection and crossover.

The mutation matrix,  $\mathcal{P}$ , has matrix elements  $\mathcal{P}_{ij} = p_m^{d^H(i,j)}(1-p_m)^{N-d^H(i,j)}$ , where  $d_{ij}^H$  is the Hamming distance between the two strings. For mutation Hamming distance is clearly a very natural metric. Note that (2) also applies for a finite population if we interpret the left hand side of (2) as the expected proportion of genotype  $c_i$  to be found at  $t+1$  while any  $P_i(t)$  on the right hand side are to be considered as the actual proportions found at  $t$ .

Explicitly  $P_c^i(t)$  is given by

$$P_{c_i}(t) = P'_i(t) + \sum_{m=1}^{2^N} \lambda_{ijk}(m) P'_j(t) P'_k(t) \quad (3)$$

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<sup>2</sup>This is not a restriction. The extension to non-fixed length strings and trees has been considered by Poli and collaborators in the context of Genetic Programming (see [8] for a recent exposition).

where  $\lambda_{ijk}(m)$  is an interaction term between strings, that depends on the particular crossover mask  $m$ , and  $\sum_{m=1}^{2^N}$  is the sum over all possible recombination masks,  $m \in M$ ,  $M$  being the space of masks. Generically,  $\lambda_{ijk}$  can be divided into two terms, one associated with string destruction,  $\lambda_{ijk}^d$ , and the other,  $\lambda_{ijk}^c$ , associated with string construction. Taking as target the string 111, an example of the former is  $111 + 000 \rightarrow 110 + 001$  while for the latter an example is the inverse of this process. To write these processes more explicitly we denote the set of bits inherited by an offspring from parent  $c_j$  as  $\mathcal{S}$  and the bits inherited from parent  $c_k$ , i.e. the set  $c_k - \mathcal{S}$ , by  $\mathcal{C}$ . Naturally,  $\mathcal{S}$  and  $\mathcal{C}$  both depend on the particular crossover mask chosen. Then,

$$\lambda_{ijk}^d = -p_c(m)\delta_{ik}(1 - \delta_{ij})\mathcal{C}_{c_i c_j}^{(1)}(m) = -p_c(m)\delta_{ik}(1 - \delta_{ij})\theta(d_{\mathcal{S}}^H(i, j))\theta(d_{\mathcal{C}}^H(i, j)) \quad (4)$$

and

$$\lambda_{ijk}^c = p_c(m)(1 - \delta_{ij})(1 - \delta_{ik})\mathcal{C}_{c_j c_k}^{(2)}(m) = \frac{p_c(m)}{2}(1 - \delta_{ij} - \delta_{ik} - \delta_{ij}\delta_{ik})[\delta(d_{\mathcal{S}}^H(i, j))\delta(d_{\mathcal{C}}^H(i, k)) + \delta(d_{\mathcal{C}}^H(i, j))\delta(d_{\mathcal{S}}^H(i, k))] \quad (5)$$

where  $p_c(m)$  is the probability to implement the mask  $m$  and the coefficients  $\mathcal{C}_{c_i c_j}^{(1)}(m)$  and  $\mathcal{C}_{c_j c_k}^{(2)}(m)$ , represent the probabilities that, given that  $c_i$  was one of the parents, it is destroyed by the crossover process, and the probability that given that neither parent was  $c_i$  it is created by recombination.  $d_{\mathcal{S}}^H(i, j)$  is the Hamming distance between the strings  $c_i$  and  $c_j$  measured only over the set  $\mathcal{S}$ , with the other arguments in (4) and (5) being similarly defined. Note that  $\mathcal{C}_{c_i c_j}^{(1)}(m)$  and  $\mathcal{C}_{c_j c_k}^{(2)}(m)$  are properties of the crossover process itself and therefore population independent. It is clear that for recombination Hamming distance is not a natural metric. For example, consider two parent strings 111111111 and 000000000. A one-point crossover implemented between the last two bits leads to offspring 111111110 and 000000001 which are Hamming distance one from the respective parents. An equally probable crossover between the fifth and six bits however, leads to 111100000 and 000001111, which are Hamming distance five away from the parents. For a given  $i$ ,  $\lambda_{ijk}$  is a  $2^N$ -dimensional matrix, but is very sparse, there being only  $O(2^N)$  non-zero elements. Thus, the microscopic representation is very inefficient, there being very few ways of creating a given target by recombination of strings. The vast majority of string recombination events are neutral in that they lead to no non-trivial interaction.

The equations (2) and (3) yield an exact expression for the probability distribution governing the evolution for arbitrary selection, mutation and crossover. It takes into account exactly the effects of destruction and construction of strings.

### 3 Coarse-Grained Evolution Equations

The dynamics of the previous section is described by  $2^N$  coupled, non-linear difference equations representing the microscopic degrees of freedom, i.e. the strings themselves. In the absence of recombination, the equations are essentially linear and, as is well known, the resulting selection/mutation problem can be recast in the guise of a two-dimensional, inhomogeneous statistical mechanics problem, where powerful techniques such as the transfer matrix approach can be invoked. However, save in very simple problems, such as a linear fitness landscape, even this simpler problem is formidable. Recombination adds an extra layer of complexity. Naturally, in such problems one always wishes to find the correct effective degrees of freedom so as to be able

to affect an effective reduction in the dimensionality of the problem. Such reductions can sometimes come about in a relatively trivial fashion, for instance, if there is an underlying symmetry that is preserved by the action of the genetic operators. This occurs for instance with selection and the genotype-phenotype map. As fitness acts at the phenotypic level then a natural coarse graining from genotypes to phenotypes occurs. This symmetry is not necessarily preserved by the other genetic operators. As a concrete example consider a fitness landscape where the fitness is given by the number of ones on the string (a simple paramagnet). In this case the dynamics can be rewritten in terms of the  $N$  phenotypes rather than  $2^N$  genotypes. The equation of motion for selection only is then

$$P_n(t+1) = \frac{n}{\bar{n}(t)} P_n(t) \quad (6)$$

where we denote phenotypes by  $n$ , the number of ones, and  $\bar{n}(t)$  is the average number of ones in the population at time  $t$ . The solution of these  $n$  difference equations is

$$P_n(t) = \frac{n^t P_n(0)}{\sum_{n=0}^N n^t P_n(0)} \quad (7)$$

Another example is that of the Eigen model, where the fitness landscape is degenerate for all genotypes except one, the master sequence. At the level of selection only, given that there are only two phenotypes, there is a reduction in the size of the configuration space from  $2^N$  to 2, i.e. a reduction in the number of degrees of freedom from  $N$  to 1. However, if we include in the effect of mutation we see there is an induced breaking of the genotype-phenotype symmetry due to the fact that strings close to the master sequence in Hamming distance have a higher “effective” fitness [9].

As mentioned in the previous section, the string representation for recombination is very inefficient due to the sparsity of the interaction matrix. This is an indication that strings are not the natural effective degrees of freedom for recombination. So what are? To form the string 111 with a recombination mask 100 one can join strings 111, 110, 101 and 100 with either 111 or 011. In other words, for the first parent the second and third bit values are unimportant and for the second the first bit value is unimportant. Thus, it is natural to coarse grain over those strings that give rise to the desired target for a given mask. Such coarse-grained variables are known as “schemata”, which we will denote by  $\xi_i$ , and are equivalent to, for instance, “block spins” in traditional statistical mechanics RG applications. The marginal probability,  $\bar{P}_i(t)$ , represents the probability of finding the schema  $\xi_i$  at time  $t$ . A specific schema is determined by summing over those bit positions that are not part of the schema. One may denote such a bit position by a \*. Thus, 11\* represents the two strings 111 and 110. The number of definite bits of the schema defines its order,  $N_2$ , while the distance between the outermost defining bits defines its length. Thus, \*11\*0\*\* has  $N_2 = 3$  and  $l = 5$ .

As there exist  $3^N$  possible schemata a full schemata basis is overcomplete as well as being non-orthonormal and corresponds to the space of all possible blocked spins. However, the space of schemata is not the natural one for recombination as we shall see. If one picks arbitrarily a vertex in  $\mathcal{G}$ , associated with a string  $c_i$ , one may perform a linear coordinate transformation  $\Lambda : \mathcal{G} \rightarrow \tilde{\mathcal{G}}$  to a basis consisting of all schemata that contain  $c_i$ . For instance, for two bits  $\mathcal{G} = \{11, 10, 01, 00\}$ , while  $\tilde{\mathcal{G}} = \{11, 1*, *1, **\}$ . The invertible matrix  $\Lambda$  is such that  $\Lambda_{ij} = 1 \iff c_j \in \xi_i$ . We denote the associated coordinate basis the Building Block basis (BBB). The BBB is complete but clearly not orthonormal. Note that the vertex  $c_i$  by construction is a fixed

point of this transformation. Apart from the vertex  $c_i$ , note that points in  $\tilde{\mathcal{G}}$  correspond to higher dimensional objects in  $\mathcal{G}$ . For instance,  $1*$  and  $*1$  are one-planes in  $\mathcal{G}$  while  $**$  is the whole space. In the BBB one finds

$$\tilde{P}_i^c(t+1) = \tilde{P}_i'(t) + \sum_{m=1}^{2^N} \tilde{\lambda}_{ijk}(m) \tilde{P}_j'(t) \tilde{P}_k'(t) \quad (8)$$

where  $\tilde{\lambda}_{ijk}(m) = \Lambda_{ii'} \lambda_{i'j'k'} \Lambda_{jj'}^{-1} \Lambda_{kk'}^{-1}$ .  $\tilde{\lambda}_{ijk}(m)$  has the property that for a given mask only interactions between BBs that construct the target schema are non-zero. i.e.  $\tilde{\lambda}_{ijk}(m) = 0$ , unless  $k$  corresponds to a schema which is the complement of  $j$  with respect to  $i$ . For example, for two bits, if we choose as vertex  $11$ , then  $11$  may interact only with  $**$ , while  $1*$  may interact only with  $*1$ . In  $\mathcal{G}$  this has the interesting interpretation that for a target schema  $\xi$  of dimensionality  $(N-d)$  only geometric objects “dual” in the  $d$ -dimensional subspace of  $\mathcal{G}$  that corresponds to  $\xi$  may interact. i.e. a  $k$ -dimensional object recombines only with a  $(N-d-k)$ -dimensional object. Additionally, a  $(N-d)$ -dimensional object may only be formed by the interaction of higher dimensional objects. In this sense interaction is via the geometric intersection of higher dimensional objects. For example, the point  $11$  can be formed by the intersection of the two lines  $1*$  and  $*1$ . Similarly,  $1111$  can be formed via intersection of the three-plane  $1***$  with the line  $*111$  or via the intersection of the two two-planes  $11**$  and  $**11$ .

Given that the object dual to a vertex is always the trivial schema,  $*...*$ , where all bits are coarse grained, and  $P(*...*, t) = 1$ , then it is instructive to combine the term linear in  $P_i'(t)$  with it's pure selection counterpart. One obtains for an arbitrary string  $c_i$

$$P_c(c_i, t) = (1 - p_c) P'(c_i, t) + \sum_{m=1}^{2^N} p_c(m) P'(c_i^S(m), t) P'(c_i^C(m), t) \quad (9)$$

where  $p_c = \sum_m p_c(m)$  and we have returned to a less abstract notation.  $P'(c_i^S(m), t)$  is the probability to select the BB  $c_i^S(m)$  and  $P'(c_i^C(m), t)$  the probability to select the BB  $c_i^C(m)$ . Both  $c_i^S(m)$  and  $c_i^C(m)$  are elements of the BBB. The above equation clearly shows that recombination is most naturally considered in terms of the BBB. The  $(2^N - 1)$  destruction terms associated with  $\lambda_{ijk}^d$  have been reduced to only one term while the  $(2^N - 1)^2$  construction terms have also been reduced to one term. Of course, we must remember that the coarse grained averages of  $c_i^C(m)$  and  $c_i^S(m)$  contain  $2^N$  terms, still, the reduction in complication is enormous. Thus, we see that recombination as an operator naturally introduces the idea of a coarse graining.

Inserting (9) in (2) we can then try to solve for the dynamics. However, in order to do that we must know the time dependence of the BB schemata  $c_i^C(m)$  and  $c_i^S(m)$ . Although the number of BB basis elements is  $2^N$  we may generalize and consider the evolution of an arbitrary schema,  $\xi$ . To do this we need to sum with  $\sum_{C_i \supset \xi}$  on both sides of the equation (2). This can simply be done to obtain [3–5] again the form (2), where this time the index  $i$  runs only over the  $2^{N_2}$  elements of the schema partition and where again  $\mathcal{P}_{ij} = p_m^{d^H(i,j)} (1 - p_m)^{N-d^H(i,j)}$ . In this case however  $d_{ij}^H$  is the Hamming distance between the two schemata. For instance, for three bit strings the schemata partition associated with the first and third bits is  $\{1*1, 1*0, 0*1, 0*0\}$ . In this case  $d_{12}^H = 1$  and  $d_{14}^H = 2$ .  $P_c(\xi, t) = \sum_{C_i \supset \xi} P_c(c_i, t)$  is the probability of finding the schema  $\xi$  after selection and crossover. Note the form invariance of the equation after coarse graining. To complete the transformation to schema dynamics we need the schema analog of (9).

This also can be obtained by acting with  $\sum_{C_i \supset \xi}$  on both sides of the equation. One obtains

$$P_c(\xi, t) = (1 - p_c \frac{N_{M_r}(\xi)}{N_M}) P'(\xi, t) + \sum_{m \in M_r} p_c(m) P'(\xi^c(m), t) P'(\xi^s(m), t) \quad (10)$$

where  $\xi^s(m)$  represents the part of the schema  $\xi$  inherited from the first parent and  $\xi^c(m)$  that part inherited from the second.  $N_{M_r}(\xi)$  is the number of crossover masks that affect  $\xi$ ,  $M_r$  being the set of such masks.  $N_M$  is the total number of masks with  $p_c(m) \neq 0$ . Obviously, these quantities depend on the type of crossover implemented and on properties of the schema such as defining length.

Thus, we see that the evolution equation for schemata is form invariant there being only a simple multiplicative renormalization of the crossover probability  $p_c$ . This form invariance, first shown in [3], demonstrates that BB schemata in general are a preferred set of coarse grained variables and more particularly the BBB is a preferred basis in the presence of recombination. It has also been shown [10] that schemata, more generally, are the only coarse graining that leads to invariance in the presence of mutation and recombination.

Considering again the structure of (9) and (10) we see that variables associated with a certain degree of coarse graining are related to BB “precursors” at an earlier time, which in their turn ... etc. This hierarchical structure terminates at order one BBs as these are unaffected by crossover. Thus, for example, the level one BB combinations of 111, i.e. BBs that lead directly upon recombination to 111 are:  $11* : **1$ ,  $1*1 : *1*$  and  $1** : *11$ . The level two BBs are  $1**$ ,  $*1*$  and  $**1$ . Thus a typical construction process is that BBs  $1**$  and  $*1*$  recombine at  $t = t_1$  to form the BB  $11*$  which at some later time  $t_2$  recombines with the BB  $**1$  to form the string 111.

#### 4 Renormalization Group

In the previous section we saw that coarse grained variables arise very naturally in genetic dynamics and gave as examples the genotype-phenotype map and schemata. We can formalize these considerations by formally introducing a general coarse graining operator  $\mathcal{R}(\eta, \eta')$  which coarse grains from the variable  $\eta$  to the variable  $\eta'$ . In this case

$$\mathcal{R}(\eta, \eta') P(\eta, t) = P(\eta', t) \quad \mathcal{R}(\eta, \eta'') P(\eta, t) = P(\eta'', t) \quad (11)$$

However, given that  $\mathcal{R}(\eta', \eta'') P(\eta', t) = P(\eta'', t)$  we deduce that

$$\mathcal{R}(\eta, \eta'') = \mathcal{R}(\eta, \eta') \mathcal{R}(\eta', \eta'') \quad (12)$$

i.e. the space of coarse grainings has a semi-group structure. Thus, we see that one can naturally introduce the RG into the study of genetic dynamics. The naturalness of a particular RG transformation will be to a large extent determined by how the dynamics looks under this coarse graining. Considering (1) for the pdf of the dynamics then given that  $\mathcal{R}(\eta, \eta') P(\eta, t) = P(\eta', t)$  the dynamics under a coarse graining is governed by  $\mathcal{R}(\eta, \eta') \mathcal{H}_\eta P(\eta, t)$ , where  $\mathcal{H}_\eta$  is the dynamical operator associated with the variables  $\eta$ . If this can be written in the form  $\mathcal{H}_{\eta'} P(\eta', t)$  with suitable renormalizations then the dynamics is form covariant or invariant under this coarse graining. As we have seen, for selection only the dynamics is invariant<sup>3</sup> when passing from genotypic to phenotypic variables, while for schemata the whole dynamics is form invariant, although

<sup>3</sup>In this case it is strictly invariant not just form invariant as there is no renormalization necessary of any parameter or variable.

there is a non-trivial renormalization of the fitness landscape as well as a simple renormalization of the recombination probability. In the case of recombination note also that the coarse graining operator associated with the BBs satisfies

$$\mathcal{R}(\eta, \eta') = \mathcal{R}(\eta^S, \eta'^S) \mathcal{R}(\eta^C, \eta'^C) \quad (13)$$

where  $\mathcal{R}(\eta^S, \eta'^S)$  represents the action of the coarse graining on the BB  $\mathcal{S}$  while  $\mathcal{R}(\eta^C, \eta'^C)$  represents the action on the BB  $\mathcal{C}$ .

## 5 Results from the coarse grained formalism

One of the strengths of the present coarse grained formulation is that much can be deduced simply by inspection of the hierarchical structure of the basic formulas. Introducing a  $2^N$ -dimensional population vector,  $\mathbf{P}(t)$ , whose elements are  $P(c_i, t)$ ,  $i = 1, \dots, 2^N$ , equation (2) can then be written in the form

$$\mathbf{P}(t+1) = \overline{\mathbf{W}}_s(t) \mathbf{P}(t) + \sum_{m=1}^{2^N} p_c(m) \overline{\mathbf{W}} \mathbf{j}(m, t) \quad (14)$$

where the selection-crossover destruction-mutation matrix  $\overline{\mathbf{W}}_s(t) = \overline{\mathcal{P}} \overline{\mathbf{F}}(t)$ . The selection - crossover destruction matrix,  $\overline{\mathbf{F}}(t)$ , is diagonal, and takes into account selection and the destructive component of crossover. Explicitly, for proportional selection  $\overline{\mathbf{F}}_{ii}(t) = (f(c_i)/\bar{f}(t))(1 - p_c)$ . Finally, the “source” matrix is given by  $\mathbf{j}(m, t) = P'(c_i^S(m), t) P'(c_i^C(m), t)$ . The interpretation of this equation is that  $\mathbf{j}(m, t)$  is a source which creates strings (or schemata) by bringing BBs together, while the first term on the right hand side tells us how the strings themselves are propagated into the next generation, the destructive effect of crossover renormalizing the fitness of the strings.

As shown in [3–5], compared to a representation based on (3), even a formal solution of (9) in the absence of mutation and for 1-point crossover and proportional selection yields much valuable qualitative information, such as a simple proof of Geiringer’s theorem and an extension of it to the weak selection regime. Explicitly, we have

$$\mathbf{P}(t) = \prod_{n=0}^{t-1} \overline{\mathbf{W}}_s(n) \mathbf{P}(0) + \sum_{m=1}^{2^N} p_c(m) \sum_{n=0}^{t-1} \prod_{i=n+1}^{t-1} \overline{\mathbf{W}}_s(i) \overline{\mathcal{P}} \mathbf{j}(m, n) \quad (15)$$

Due to the form invariance of the equations this solution actually holds true for arbitrary schemata. The only changes are that the vectors are of dimension  $2^{N_2}$ , the matrices of dimension  $2^{N_2} \times 2^{N_2}$ , the sum over masks for the construction terms is only over the set  $M_r$  and that the BBs in  $\mathbf{j}(m, t)$  are those of the schema rather than the entire string.

The interpretation of (15) follows naturally from that of (14). Considering first the case without mutation, the first term on the right hand side gives us the probability for propagating a string or schema from  $t = 0$  to  $t$  without being destroyed by crossover. In other words  $\prod_{n=0}^{t-1} \overline{\mathbf{W}}_s(n)$  is the Greens function or propagator for  $\mathbf{P}$ . In the second term,  $\mathbf{j}(m, n)$ , each element is associated with the creation of a string or schema at time  $n$  via the juxtaposition of two BBs associated with a mask  $m$ . The factor  $\prod_{i=n}^{t-1} \overline{\mathbf{W}}_s(i)$  is the probability to propagate the resultant string or schema without crossover destruction from its creation at time  $n$  to  $t$ . The sum over masks and  $n$  is simply the sum over all possible creation events in the dynamics.

This formulation lends itself very naturally to a diagrammatic representation and the formulation of a set of “Feynman rules” which allow for the calculation of  $P(\xi, t)$  in the BBB. Here for simplicity and transparency we write them for  $p_m = 0$ . The generalization to “matrix” propagators in the presence of mutation is straightforward.

- 1) Draw all possible connected tree diagrams that contribute to  $\xi$
- 2) For each diagram to each internal line attach a propagator

$$F_{ij}(t, t') = (1 - p_c \frac{N_{M_r}(\xi_j)}{N_M})^{t-t'} \prod_{n=t'}^t \delta_{ij} \frac{f(\xi_j, n)}{f(n)}$$

- 3) To each vertex assign a weight

$$\tilde{\lambda}_{ijk} = p_c(m) \frac{f(\xi_j^S(m), n)}{\bar{f}(n)} \frac{f(\xi_k^C(m), n)}{\bar{f}(n)} \delta(d_k + d_i - (N - d_j))$$

- 4) Carry out the integration over time for all vertices

In the above  $d_i$  represents the dimensionality of the schema  $i$ . As a simple example consider two bit strings and  $p_c = 1$ . Consider the pdf for 11. In this case there are only two diagrams; the diagram corresponding to propagation of 11 itself from  $t = 0$  to  $t$  and the formation of 11 by recombination of 1\* and \*1 at time  $t'$ ,  $0 \leq t' < t$ . In this case  $F_{11}(t, 0) = 0$ ,  $F_{1*}(t, 0) = \prod_{n=0}^{t-1} (f(1*, n)/\bar{f}(n))$  and similarly for \*1. For the interactions the only non-zero vertex is  $\tilde{\lambda}_{11,1*,*1} = (f(1*, n)/\bar{f}(n))(f(*1, n)/\bar{f}(n))$ . The diagrammatic series is naturally a perturbation series in the number of BB recombination events. In the case of a flat fitness landscape the entire diagrammatic series can be exactly resummed for an arbitrary string in the continuous time limit to find [6]

$$P(C_i, t) = \sum_{n=0}^{N-1} e^{-\frac{np_c t}{N-1}} (1 - e^{-\frac{p_c t}{N-1}})^{N-n-1} \mathcal{P}(n+1) \quad (16)$$

where  $\mathcal{P}(n+1)$  is an initial condition and represents a partition over the probabilities for finding  $(N - n)$  building blocks at  $t = 0$ . For a given  $n$  there are  $^{N-1}C_n$  such terms. For instance, for 11,  $\mathcal{P}(2) = P(11, 0)$  and  $\mathcal{P}(1) = P(1*, 0)P(*1, 0)$ . Note that the simple dynamical form arises because of the use of the BBB. One can use  $\Lambda$  to rewrite the above result in a string basis. The resulting expression is far more complicated in that the dynamical factors for a given string combination are complicated combinations of those associated with the BBB. Note also that the fixed point is non-perturbative in  $p_c$  indicating that the asymptotic dynamics cannot be accessed perturbatively. This is why it was necessary to sum the entire diagrammatic series.

The tendency of recombination is to destroy correlations between different loci in the population. Selection, depending on the landscape, can induce correlations, hence there is a competition. In the case of weak selection and strong crossover (16) shows that correlations asymptotically decay and hence the effective degrees of freedom are 1-schemata. Higher order schemata can be taken as perturbations around this decorrelated limit. In the case of a recombination operator that mixes freely all bits within the entire population (genepool recombination) then these perturbations are zero and the 1-schemata give an exact description of the dynamics. Under



these circumstances one may solve the dynamics exactly, including mutation, for certain fitness landscapes, such as a linear fitness landscape [7].

## 6 Conclusions

We have here briefly tried to lay out why RG concepts and techniques can be useful when studying the dynamics of genetic systems. I showed that genetic dynamics can be profitably studied in the context of coarse grained degrees of freedom. What is a natural coarse graining was seen to depend on the genetic operators present. We saw that for recombination the equations of motion were form invariant under a schemata coarse graining and that the BBB was a preferred one leading to a recombination dynamics where effective degrees of freedom are related to more coarse grained BBs. We showed that the hierarchical nature of recombination led to a natural formulation in terms of Feynman diagrams with an associated set of Feynman rules. We also briefly mentioned some concrete results that emerge naturally from a coarse grained formulation. We strongly believe that the RG has an important role to play in developing a more quantitative understanding of the dynamics of genetic systems and hope there will be interesting developments to report at the next RG conference.

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